

# THE STRUCTURE AND SPECTRUM OF HEISENBERG ODOMETERS

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**ABSTRACT.** In [3] the authors define odometer actions of discrete, finitely generated and residually finite groups  $G$ . In this paper we focus on the case where  $G$  is the discrete Heisenberg group. We prove a structure theorem for finite index subgroups of the Heisenberg group based on their geometry when they are considered as subsets of  $\mathbb{Z}^3$ . We provide a complete classification of Heisenberg odometers based on the structure of their defining subgroups and we provide examples of each class. It follows from [7] that all such actions have discrete spectrum, i.e. that the unitary operator associated to the dynamical system admits a decomposition into finite dimensional, irreducible representations of the group  $G$ . Here we provide an explicit proof of this fact for general  $G$  odometers. Our proof allows us to define explicitly those representations of the Heisenberg group which appear in the spectral decomposition of a Heisenberg odometer, as a function of the defining subgroups. Along the way we also provide necessary and sufficient conditions for a  $\mathbb{Z}^d$  odometer to be a product odometer as defined by Cortez in [2].

## 1. INTRODUCTION

Odometer systems are a well studied class of examples in the classical theory of measurable and topological dynamical systems generated by a single transformation. They are rank one transformations, and therefore are ergodic and have zero entropy. They have discrete rational spectrum and are the key ingredients in the study of Toeplitz systems. They can be viewed measure theoretically as cutting and stacking transformations of the unit interval. Alternatively, they can be viewed algebraically as an action of  $\mathbb{Z}$  on an inverse limit space of increasing quotient groups of  $\mathbb{Z}$ . They can also be viewed as an action by addition in an adic group. It follows that they are, in fact uniquely ergodic (see, for example, [4] and the references therein, and [8]).

All of these perspectives can be generalized to define odometer actions of  $\mathbb{Z}^d$  and there is an obvious way to construct examples. In the case of  $d = 2$ , given any two  $\mathbb{Z}$  odometer actions  $(T, X)$  and  $(S, Y)$ , the maps  $T \times Id, Id \times S$  acting on  $X \times Y$  clearly commute and satisfy the appropriate generalizations of the above ideas to  $\mathbb{Z}^2$ . In [2] Cortez defines odometer actions of  $\mathbb{Z}^d$  using the inverse limit approach. She calls the obvious examples described above *product odometers* and gives an example of a non-product type  $\mathbb{Z}^2$  odometer. As in the classical case,  $\mathbb{Z}^d$  odometers are uniquely ergodic and have zero entropy. In [3] Cortez and Petite generalize the work in [2] to define  $G$  odometer dynamical systems for  $G$  any discrete, finitely

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generated and residually finite group, and show that they are also uniquely ergodic. In this paper we provide a detailed geometric analysis of the special case of odometer actions of the discrete Heisenberg group and a spectral analysis of  $G$  odometers in general. Our work allows us to give an explicit description of the finite dimensional, irreducible representations of the Heisenberg group which can arise in the spectral decomposition of a Heisenberg odometer.

**Heisenberg odometers.** Let  $H$  be the discrete Heisenberg group, defined on the set  $\mathbb{Z}^3$  with the following group multiplication:

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy').$$

In this paper we show that there are geometric considerations similar to the  $\mathbb{Z}^2$  case which separate different types of Heisenberg odometers. In particular, thinking of subgroups of the Heisenberg group as subsets of  $\mathbb{Z}^3$  with a different group multiplication, we can associate to any Heisenberg odometer, a  $\mathbb{Z}^2$  odometer constructed by considering the projections of the Heisenberg subgroups onto their first two coordinates. If this associated odometer is of product type, as defined by Cortez, we call the Heisenberg odometer an  $(x, y)$ -product odometer. If the subgroups  $\Gamma_n$ , considered as sets in  $\mathbb{Z}^3$ , have the structure  $A\mathbb{Z}^2 \times m\mathbb{Z}$  for some nonsingular matrix  $A \in M(2, \mathbb{Z})$  and  $m \in \mathbb{N}$  we call it a *flat* odometer. If the Heisenberg odometer is both of  $(x, y)$ -product type and flat we call it a *pure product* odometer. As in the  $\mathbb{Z}^d$  case, it is obvious how to construct pure product type odometers for the Heisenberg group. We show that these are not the only odometers one can construct. In fact, we show that there are odometers of all possible combinations and that these are the only classes possible (see Figure 1 for a guide to the examples constructed in the paper).

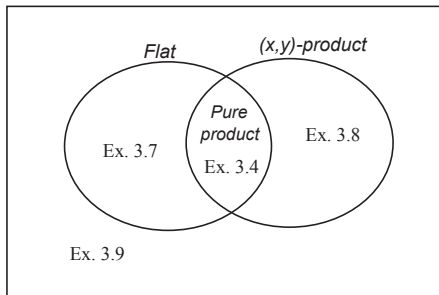


FIGURE 1. Examples of all possible classes of Heisenberg odometers

In order to construct Heisenberg odometers that are not of product type we also extend the work in [2]. We give necessary and sufficient conditions for a  $\mathbb{Z}^d$  odometer to be of product type. Our characterization allows us to identify a larger collection of non-product examples than the work in [2] would allow.

**Spectral analysis of  $G$  odometers.** Let  $G$  be as described above and recall that an ergodic and measure preserving action of  $G$  is said to have discrete spectrum if the associated unitary representation can be decomposed into a direct sum of irreducible, finite dimensional representations of  $G$ . In [7] Mackey shows that any action of this class of groups  $G$  which is conjugate to a rotation on a compact

group by a dense subgroup has discrete spectrum as a consequence of the Peter-Weyl Theorem. A  $G$  odometer is an example of this type of action. Here we present a different argument where we explicitly construct the decomposition into irreducible finite dimensional representations of  $G$ . As an easy corollary, for those groups  $G$  where entropy theory has been sufficiently developed, we have that  $G$  odometers have zero entropy.

In the case of Heisenberg odometers, our analysis of the geometric structure of the subgroups of  $H$  allows us to give a complete description of the finite dimensional, irreducible representations of  $H$  that can occur in the spectral decomposition of any given Heisenberg odometer action.

## 2. DEFINING $G$ ODOMETERS

Let  $G$  be a discrete, finitely generated and residually finite group. Following [3] we define a  $G$  odometer dynamical system as follows. Since  $G$  is residually finite, there exists a sequence  $\Gamma_1 \supset \Gamma_2 \supset \cdots \Gamma_n \supset \cdots$  of subgroups with finite indices in  $G$  such that  $\cap \Gamma_n = \{e\}$ . Let  $\pi_n: G/\Gamma_{n+1} \rightarrow G/\Gamma_n$  be the homomorphism induced by the inclusion  $\Gamma_{n+1} \subset \Gamma_n$  and denote by  $\overleftarrow{G/\Gamma_n}$  the inverse limit space of the sequence  $\{(G/\Gamma_n, \pi_n)\}_{n \geq 1}$ . It is a compact metrizable space whose topology is spanned by the cylinder sets

$$[n; \gamma] = \{g \in \overleftarrow{G/\Gamma_n} : g_n = \gamma\} \text{ with } \gamma \in G/\Gamma_n.$$

The group  $G$  acts by left multiplication on  $\overleftarrow{G/\Gamma_n}$  and we denote the action by  $h.g$ . Such an action is called a  $G$  subodometer. If the subgroups  $\{\Gamma_n\}$  are normal in  $G$ , the system is called a  $G$  odometer. The action of  $G$  on  $\overleftarrow{G/\Gamma_n}$  preserves Haar measure  $\mu$  and is uniquely ergodic. We refer to the dynamical system  $(\overleftarrow{G/\Gamma_n}, \mu, G)$  as the  $G$  odometer on  $\overleftarrow{G/\Gamma_n}$ .

## 3. SUBGROUPS OF THE DISCRETE HEISENBERG GROUP

In this section we describe the geometry of subgroups of  $H$ , the discrete Heisenberg group, in terms of their projection onto their first two coordinates, and the structure of the fiber over this projection. The work in this section will allow us to carry out the classification of Heisenberg odometers described in the introduction.

The following facts are easily verified by computation and we will use them frequently in our arguments:

$$\begin{aligned} (x, y, z)^{-1} &= (-x, -y, -z + xy) \\ (1) \quad (u, v, w)(x, y, z)(u, v, w)^{-1} &= (x, y, z + uy - vx) \\ (u, v, w)(x, y, z)(u, v, w)^{-1}(x, y, z)^{-1} &= (0, 0, uy - vx). \end{aligned}$$

**Proposition 3.1.** *Any finite index subgroup  $\Gamma$  of  $H$  can be written as*

$$(2) \quad \Gamma = \{(x, y, i_\Gamma(x, y) + km_\Gamma) : (x, y) \in A\mathbb{Z}^2, k \in \mathbb{Z}\}$$

for some nonsingular matrix  $A \in M(2, \mathbb{Z})$ ,  $m_\Gamma \in \mathbb{N}$  with  $m_\Gamma \geq 1$  and  $i_\Gamma : A\mathbb{Z}^2 \rightarrow \mathbb{Z}$  defined by

$$i_\Gamma(x, y) = \inf\{z \geq 0 : (x, y, z) \in \Gamma\}.$$

*Proof.* Let  $f$  and  $g$  be the group homomorphisms from the following short exact sequence:

$$0 \rightarrow \mathbb{Z} = \langle (0, 0, 1) \rangle \xrightarrow{f} H \xrightarrow{g} \mathbb{Z}^2 \rightarrow 0.$$

Define  $f_\Gamma = f|_{\langle (0,0,1) \rangle \cap \Gamma}$  and  $g_\Gamma = g|_\Gamma$ . Then the following is also a short exact sequence:

$$0 \rightarrow \langle (0, 0, 1) \rangle \cap \Gamma \xrightarrow{f_\Gamma} \Gamma \xrightarrow{g_\Gamma} A\mathbb{Z}^2 \rightarrow 0$$

where  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{Z})$  is such that  $A\mathbb{Z}^2 = \text{Im } g_\Gamma$ . We then have that

$$(3) \quad A\mathbb{Z}^2 \sim \Gamma / \ker g_\Gamma \sim \Gamma / \text{Im } f_\Gamma \sim \Gamma / m_\Gamma \mathbb{Z}$$

for some non-negative integer  $m_\Gamma$ , and it follows that

$$(4) \quad \ker g_\Gamma = \{(0, 0, km_\Gamma) : k \in \mathbb{Z}\}.$$

Since  $\Gamma$  has finite index it follows immediately that  $m_\Gamma \geq 1$  and that  $A$  is nonsingular. Further, let  $(x, y) \in A\mathbb{Z}^2$  and suppose  $(x, y, z)$  and  $(x, y, z')$  are both elements of  $\Gamma$ . Then (3) and (4) shows that  $z = z' \pmod{m_\Gamma}$  concluding the proof.  $\square$

In order to construct Heisenberg odometers we will only be considering subgroups  $\Gamma$  with the property that  $m_\Gamma > 1$ . The next proposition identifies some algebraic properties of the function  $i_\Gamma$ , and the relationship between the constant  $m_\Gamma$  and the matrix  $A$ .

**Proposition 3.2.** *Let  $\Gamma$  be a finite index subgroup of  $H$ , and  $A, i_\Gamma, m_\Gamma$  be as in Proposition 3.1. Then*

$$(5) \quad i_\Gamma(x, y) + i_\Gamma(x', y') + xy' = i_\Gamma(x + x', y + y') \pmod{m_\Gamma}$$

and

$$(6) \quad m_\Gamma | \det(A).$$

If, in addition,  $\Gamma$  is a normal subgroup then we have the stronger conclusion that  $m_\Gamma$  divides all the entries in the matrix  $A$ .

*Proof.* Let  $(x, y, i_\Gamma(x, y) + km_\Gamma), (x', y', i_\Gamma(x', y') + k'm_\Gamma) \in \Gamma$ . Proposition 3.1 yields that

$$i_\Gamma(x, y) + i_\Gamma(x', y') + (k + k')m_\Gamma + xy' = i_\Gamma(x + x', y + y') + k''m_\Gamma$$

for some  $k'' \in \mathbb{Z}$ , and (5) follows. To see that (6) holds choose  $p, q, r, s \in \mathbb{Z}$  such that  $m_\Gamma$  is relatively prime to  $ps - rq$ . Let

$$\begin{pmatrix} u & x \\ v & y \end{pmatrix} = A \begin{pmatrix} p & r \\ q & s \end{pmatrix}.$$

Then both  $(u, v), (x, y) \in A\mathbb{Z}^2$  and therefore there exist  $w, z \in \mathbb{Z}$  such that  $(u, v, w), (x, y, z) \in \Gamma$ . Using (1) we then have that  $(0, 0, ux - vy) \in \Gamma$  and (4) implies that  $m_\Gamma$  must divide  $ux - vy$ . But

$$uy - vx = \det \begin{pmatrix} u & x \\ v & y \end{pmatrix} = \det(A) \det \begin{pmatrix} p & r \\ q & s \end{pmatrix}.$$

so by our choice of  $p, q, r, s$  we must have that  $m_\Gamma$  divides  $\det(A)$ .

Finally, suppose that  $\Gamma$  is normal, and choose any  $(x, y, z) \in \Gamma$  and  $(u, v, w) \in G$ . Normality, the definition of  $m_\Gamma$ , and (1) yield that  $m_\Gamma | (uy - vx)$  for any  $(u, v) \in \mathbb{Z}^2$ .

This implies that  $m_\Gamma | x, y$ . Since  $(x, y) \in A\mathbb{Z}^2$  is arbitrary, it follows that  $m_\Gamma$  must divide each entry of the matrix  $A$ .  $\square$

The following proposition gives a converse result to the previous two propositions.

**Proposition 3.3.** *Consider a nonsingular matrix  $A \in M(2, \mathbb{Z})$ ,  $m > 1$ , and a map  $i : A\mathbb{Z}^2 \rightarrow \mathbb{Z}$  chosen such that  $m$  divides the entries of  $A$ , and  $i$  satisfies*

$$(7) \quad i(x, y) + i(x', y') = i(x + x', y + y') \pmod{m}.$$

Then

$$\Gamma_{A, i, m} = \{(x, y, km + i(x, y)) : (x, y) \in A\mathbb{Z}^2, k \in \mathbb{Z}\}$$

is a finite index normal subgroup of  $H$ .

*Proof.* If  $\Gamma_{A, i, m}$  is a subgroup, our choice of  $A$  and  $m$  will guarantee that it is a subgroup of finite index. To see that it is a subgroup first note that by (7) for any  $k \in \mathbb{Z}$  we have  $ki(0, 0) = i(0, 0) \pmod{m}$ . Therefore we must have  $i(0, 0) = 0$  and  $(0, 0, 0) \in \Gamma_{A, i, m}$ . Choose  $\gamma = (x, y, i(x, y) + km)$  for some  $k \in \mathbb{Z}$ . To see that  $\gamma^{-1}$  is also in  $\Gamma$  we use (1) to obtain  $\gamma^{-1} = (-x, -y, -i(x, y) - km + xy)$ . Using (7) we can replace  $-i(x, y)$  with  $i(-x, -y) + k'm$  for some  $k' \in \mathbb{Z}$ . Since  $m$  divides the entries of  $A$ , it must divide  $x$  and  $y$  yielding that  $\gamma^{-1}$  lies in  $\Gamma$ .

Closure under addition and normality follow from similar arguments.  $\square$

*Remark 3.4.* Note that  $i = i_{\Gamma_{A, i, m}} \pmod{m}$  where  $i_{\Gamma_{A, i, m}}$  is as defined in Proposition 3.1.

The next result will allow us to easily check if a sequence of normal subgroups of  $H$  is nested.

**Proposition 3.5.** *Suppose  $\Gamma$  and  $\Gamma'$  are finite index normal subgroups of  $H$  defined by the triples  $(A_\Gamma \mathbb{Z}^2, m_\Gamma, i_\Gamma)$  and  $(A_{\Gamma'} \mathbb{Z}^2, m_{\Gamma'}, i_{\Gamma'})$ , respectively. Then  $\Gamma' \leq \Gamma$  if and only if  $A_{\Gamma'} \mathbb{Z}^2 \subset A_\Gamma \mathbb{Z}^2$ ,  $m_\Gamma | m_{\Gamma'}$  and*

$$(8) \quad i_{\Gamma'}(x, y) = i_\Gamma(x, y) \pmod{m_\Gamma} \text{ for all } (x, y) \in A_{\Gamma'} \mathbb{Z}^2.$$

*Proof.* The proof follows immediately when the subgroups  $\Gamma$  and  $\Gamma'$  are written in the form given by (2) of Proposition 3.1.  $\square$

#### 4. THE CLASSIFICATION OF HEISENBERG ODOMETER ACTIONS

Recall from the introduction that it is possible to classify Heisenberg odometers in terms of the geometric structure of the defining subgroups. We will now use the results of the previous section to carry out this classification. In particular, we can now describe any sequence of subgroups  $\Gamma_n$  of  $H$  in terms of the sequence of triples  $(A_n \mathbb{Z}^2, m_{\Gamma_n}, i_{\Gamma_n})$ .

**Definition 4.1.** A finite index subgroup  $\Gamma$  of  $H$  is called:

- *flat* if  $i_\Gamma \equiv 0$ ;
- *$(x, y)$ -product* if there exists a diagonal nonsingular matrix  $A \in M(2, \mathbb{Z})$  such that  $\text{Im} \gamma_\Gamma = A\mathbb{Z}^2$ ;
- *pure product* if it is flat and  $(x, y)$ -product.

Based on this definition, we introduce the following classes of  $H$  odometers.

**Definition 4.2.** An  $H$  odometer on  $\overleftarrow{H/\Gamma_n}$  is called a *flat*  $((x, y)$ -product, pure product) odometer if it is conjugate to an  $H$  odometer on  $\overleftarrow{H/\Gamma'_n}$  where each normal subgroup  $\Gamma'_n$  is flat  $((x, y)$ -product, pure product).

There exists an effective criterion to check whether two  $H$  odometers are conjugate which we will use extensively in what follows. It is based on the following characterization of a factor map:

**Lemma 4.3** ([3]). *There exists a factor map  $\pi : \overleftarrow{H/\Gamma_n^1} \rightarrow \overleftarrow{H/\Gamma_n^2}$  between two  $H$  odometers if and only if for every  $\Gamma_n^2$  there exists  $\Gamma_k^1$  such that  $\Gamma_k^1 \subset \Gamma_n^2$ .*

Note that if the sequence  $\{\Gamma_n\}$  defines a Heisenberg odometer, then the sequence  $\{A_n\mathbb{Z}^2\}$  must also define a  $\mathbb{Z}^2$  odometer. We call the odometer on  $\overleftarrow{\mathbb{Z}^2/A_n\mathbb{Z}^2}$  the *associated  $\mathbb{Z}^2$  odometer*. It is clear that a Heisenberg odometer is of  $(x, y)$ -product type if and only if the associated  $\mathbb{Z}^2$  odometer is of product type and therefore pure product Heisenberg odometers are very easy to construct.

**Example 4.4 (Pure product Heisenberg odometer).** Let  $A_n = \begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}$ ,  $m_n = 2^n$ ,  $i_n = 0$ .

In producing Heisenberg odometers which are not  $(x, y)$ -product we can't simply use Cortez's [2] example of a non-product  $\mathbb{Z}^2$  odometer. That example is constructed with the following sequence of matrices:

$$(9) \quad \begin{pmatrix} 3^{n+1} & 7 \cdot 11^n \\ 7 \cdot 3^n & 11^{n+1} \end{pmatrix}$$

which have the property that the entries are relatively prime, a sufficient condition for the resulting odometer to be non-product type. However, by Proposition 3.2 any sequence of normal subgroups  $\Gamma_n$  of the Heisenberg group with such a projection must have  $m_{\Gamma_n} = 1$  for all  $n$  and therefore won't have trivial intersection. Thus these family of non-product type  $\mathbb{Z}^2$  examples cannot be used to construct Heisenberg odometers.

In what follows we give a new necessary and sufficient condition for a  $\mathbb{Z}^d$  odometer to be conjugate to a product odometer which allows us to give examples where the entries of the matrices have a non-trivial common factor. Examples of flat and non  $(x, y)$ -product type Heisenberg odometers will follow immediately.

**Proposition 4.5.** *A  $\mathbb{Z}^d$  odometer on  $\overleftarrow{\mathbb{Z}^d/A_n\mathbb{Z}^d}$  is conjugate to a product odometer if and only if there exists a subsequence of  $\{A_n\}$  such that*

$$(10) \quad m_k(A_n) \mid \text{row}_k(A_{n+1}) \quad \forall 1 \leq k \leq d$$

where  $m_k(A_n) = \min\{m \geq 1 : m\vec{e}_k \in A_n\mathbb{Z}^d\}$ .

*Proof.* Assume that the odometer on  $\overleftarrow{\mathbb{Z}^d/A_n\mathbb{Z}^d}$  is conjugate to the odometer on  $\overleftarrow{\mathbb{Z}^d/\Delta_n\mathbb{Z}^d}$  where each  $\Delta_n \in M(d, \mathbb{Z})$  is a nonsingular diagonal matrix. The inclusion  $\Delta_n\mathbb{Z}^d \supset \Delta_{n+1}\mathbb{Z}^d$  is equivalent to

$$\Delta_n(k, k) \mid \Delta_{n+1}(k, k) \quad \forall k = 1, \dots, d.$$

Also, by Lemma 4.3, and passing as necessary to a subsequence, one has

$$(11) \quad A_n\mathbb{Z}^d \supset \Delta_n\mathbb{Z}^d \supset A_{n+1}\mathbb{Z}^d.$$

Since  $\Delta_n \vec{e}_k = \Delta_n(k, k) \vec{e}_k \in A_n \mathbb{Z}^d$ , it follows that  $m_k(A_n) | \Delta_n(k, k)$ . From  $A_{n+1} \mathbb{Z}^d \subset \Delta_n \mathbb{Z}^d$  we also have that  $\Delta_n(k, k) | \text{row}_k(A_{n+1})$ . Therefore

$$m_k(A_n) | \Delta_n(k, k) | \text{row}_k(A_{n+1}).$$

Conversely, if (10) holds, one considers the sequence of diagonal matrices  $\{\Delta_n\}$  given by  $\Delta_n(k, k) = m_k(A_n)$ . Notice that  $\Delta_n \mathbb{Z}^d \supset A_{n+1} \mathbb{Z}^d$ . Moreover,  $\Delta_n \vec{e}_k = m_k(A_n) \vec{e}_k \in A_n \mathbb{Z}^d$ , hence  $\Delta_n \mathbb{Z}^d \subset A_n \mathbb{Z}^d$ . Therefore (11) holds, and the odometers on  $\overleftarrow{\mathbb{Z}^d / A_n \mathbb{Z}^d}$  and  $\overleftarrow{\mathbb{Z}^d / \Delta_n \mathbb{Z}^d}$  are conjugate.  $\square$

For any nonsingular matrix  $A \in M(d, \mathbb{Z})$  one can factor out the greatest common (positive) factor of each row into a diagonal matrix  $\Delta$  and write  $A = \Delta \cdot \hat{A}$  where each row of  $\hat{A} \in M(d, \mathbb{Z})$  has relatively prime entries. For  $d = 2$ , one can check that  $m_k(A)$  as defined above satisfies  $m_k(A) = \Delta(k, k) \cdot |\det(\hat{A})|$ . Thus we have the following immediate corollary:

**Corollary 4.6.** *A  $\mathbb{Z}^2$  odometer on  $\overleftarrow{\mathbb{Z}^2 / A_n \mathbb{Z}^2}$  is conjugate to a product odometer if and only if there is a subsequence of  $\{A_n\}_{n \in \mathbb{N}}$  such that*

$$(12) \quad \Delta_n(k, k) \cdot \det(\hat{A}_n) | \text{row}_k(A_{n+1}) \quad \forall 1 \leq k \leq d, n \in \mathbb{N}.$$

We are now ready to produce a flat Heisenberg odometer which is not  $(x, y)$ -product.

**Example 4.7 (Flat but not  $(x, y)$ -product type Heisenberg odometer).** We modify the matrices in (9) slightly:

$$(13) \quad A_n = \begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix} \cdot \begin{pmatrix} 3^{n+1} & 7 \cdot 11^n \\ 7 \cdot 3^n & 11^{n+1} \end{pmatrix}$$

and we consider the subgroups  $\Gamma_n$  given by the triple  $(A_n \mathbb{Z}^2, 2^n, 0)$ . By Proposition 3.3 each  $\Gamma_n$  is a normal subgroup of  $H$ . It is easy to check that  $\Gamma_{n+1} \subset \Gamma_n$  and  $\cap \Gamma_n = \{(0, 0, 0)\}$ , hence the  $H$  odometer on  $\overleftarrow{H / \Gamma_n}$  is flat. By Corollary 4.6 the associated  $\mathbb{Z}^2$  odometer is not of product type. Indeed,  $\Delta_n(1, 1) \cdot \det(\hat{A}_n) = 2^n \cdot (-16) \cdot 33^n$ , and  $A_m(1, 1) = 2^m \cdot 3^{m+1}$  and there are no  $m > n$  so that  $2^n \cdot (-16) \cdot 33^n$  divides  $2^m \cdot 3^{m+1}$ .

Given any sequence  $\{A_n \mathbb{Z}^2\}$  it is not obvious how to choose a non-trivial sequence  $i_n$  so that the resulting sequence of subgroups will give rise to a non-flat odometer, or if it is possible to do so even for product type odometers. Below we provide an example of an  $(x, y)$ -product but not flat Heisenberg odometer.

**Example 4.8 ( $(x, y)$ -product but not flat Heisenberg odometer).** Consider the sequence of positive integers  $\{k_n\}$  defined recursively by  $k_1 = 2$  and  $k_{n+1} = k_n(k_n + 1)$ . For each  $n$ , we construct by Proposition 3.2 the normal subgroup  $\Gamma_n$  using the diagonal matrix  $A_n = \begin{pmatrix} k_n & 0 \\ 0 & k_n \end{pmatrix}$ , positive integer  $m_n = k_n$  and the map  $i_n : A_n \mathbb{Z}^2 \rightarrow \mathbb{Z}$  defined as  $i_n(x, y) = x/k_n$ . Notice that

$$\Gamma_n = \{(k_n u, k_n v, k_n w + u) : u, v, w \in \mathbb{Z}\}.$$

We now check  $\Gamma_{n+1} \subset \Gamma_n$  using Proposition 3.5. The only non-trivial condition to verify is (8): if  $x = k_{n+1}u$ , then  $i_{n+1}(x, y) = u$  and  $i_n(x, y) = (k_n + 1)u = u \pmod{k_n}$ , as needed.

Also  $\cap \Gamma_n = \{(0, 0, 0)\}$ , otherwise if  $(0, 0, 0) \neq (x, y, z) \in \cap \Gamma_n$ , then at least one of  $|x|, |y|, |z| \geq k_n$  for all  $n \geq 1$ , which is impossible since  $k_n \rightarrow \infty$ . Therefore the  $H$  odometer on  $\overleftarrow{H}/\Gamma_n$  is of  $(x, y)$ -product type.

Notice that  $(k_n, k_n, 1) \in \Gamma_n$  for every  $n \in \mathbb{N}$ . If this example has an odometer factor on  $\overleftarrow{H}/\Gamma'_n$  where each subgroup  $\Gamma'_n$  is flat, then by Lemma 4.3 it would follow that for any  $\Gamma'_n$  there exists  $\Gamma_m$  such that  $\Gamma_m \subset \Gamma'_n$ , and so  $(k_m, k_m, 1) \in \Gamma'_n$ , as well. This contradicts the fact that  $\Gamma'_n$  is flat:  $i_{\Gamma'_n} \equiv 0$  and the  $z$ -component cannot be 1 (by Proposition 3.1).

**Example 4.9 (Not flat and not  $(x, y)$ -product Heisenberg odometer).** Using the sequence  $\{k_n\}$  defined in the previous example and the matrices in (9), we consider the sequence of matrices

$$(14) \quad A_n = \begin{pmatrix} k_n & 0 \\ 0 & k_n \end{pmatrix} \cdot \begin{pmatrix} 3^{n+1} & 7 \cdot 11^n \\ 7 \cdot 3^n & 11^{n+1} \end{pmatrix},$$

and the normal subgroups given by the triples  $(A_n, k_n, i_n)$  where  $i_n : A_n \mathbb{Z}^2 \rightarrow \mathbb{Z}$  is defined as  $i_n(x, y) = x/k_n$ . We can describe  $\Gamma_n$  as

$$\{(k_n(3^{n+1} \cdot u + 7 \cdot 11^n \cdot v), k_n(7 \cdot 3^n \cdot u + 11^{n+1} \cdot v), k_n \cdot w + 3^{n+1} \cdot u + 7 \cdot 11^n \cdot v) : u, v, w \in \mathbb{Z}\}.$$

The inclusion  $\Gamma_{n+1} \subset \Gamma_n$  follows from Proposition 3.5: in order to verify (8) notice that if  $x = k_{n+1}(3^{n+2}u + 7 \cdot 11^{n+1}v)$ , then  $i_{n+1}(x, y) = 3^{n+2}u + 7 \cdot 11^{n+1}v$  and  $i_n(x, y) = (k_n + 1)(3^{n+2}u + 7 \cdot 11^{n+1}v) = 3^{n+2}u + 7 \cdot 11^{n+1}v \pmod{k_n}$ , as needed.

An argument similar to that of the previous example shows that  $\cap \Gamma_n = \{(0, 0, 0)\}$ , hence the sequence  $\{\Gamma_n\}$  defines an Heisenberg odometer.

By choosing  $u, v \in \mathbb{Z}$  such that  $3^{n+1} \cdot u + 7 \cdot 11^n \cdot v = 1$  and letting  $w = 0$ , we have  $(k_n, k_n, 1) \in \Gamma_n$  for every  $n \in \mathbb{N}$ . We conclude, as above, that the odometer on  $\overleftarrow{H}/\Gamma_n$  cannot be flat.

We analyze now the associated  $\mathbb{Z}^2$  odometer,  $\overleftarrow{\mathbb{Z}^2}/A_n \mathbb{Z}^2$  and show that the condition stated in Corollary 4.6 is not satisfied. Indeed,  $\Delta_n(1, 1) \cdot \det(\hat{A}_n) = k_n \cdot (-16) \cdot 33^n$ , and  $A_m(1, 1) = k_m \cdot 3^{m+1}$  and there are no  $m > n$  so that  $k_n \cdot (-16) \cdot 33^n$  divides  $k_m \cdot 3^{m+1}$ , since each integer  $k_m = 2 \cdot \text{odd}$ .

## 5. SPECTRAL ANALYSIS OF HEISENBERG ODOMETER ACTIONS

As was discussed in the introduction, Mackey [7] has shown that a  $G$  odometer, for  $G$  any discrete, finitely generated and residually finite group, has discrete spectrum by showing that the action of  $G$  on the inverse limit space  $\overleftarrow{G}/\Gamma_n$  is a sub-action of the compact group  $\overleftarrow{G}/\Gamma_n$  acting on itself by rotation. In this section we analyze the exact nature of this decomposition.

It is clear that any one dimensional irreducible representation that appears comes from eigenfunctions of the group action. In [3] the authors show that the eigenvalues of a  $G$  odometer are those characters  $\phi : G \rightarrow S^1$  of the group  $G$  for which  $\phi(\gamma) = 1$  for all  $\gamma \in \Gamma_n$  for some  $n$  and that the functions  $f = \sum_{\gamma \in G/\Gamma_n} \phi(\gamma) \chi_{[n; \gamma]}$  are the corresponding eigenfunctions. Below we present an alternate proof that odometer actions have discrete spectrum that allows us to identify the rest of the representations that occur in the decomposition explicitly and in terms of the irreducible, finite dimensional representations of the group  $G$ , as opposed to the group  $\overleftarrow{G}/\Gamma_n$ .



In the case of the discrete Heisenberg group  $H$  this approach allows us to say much more. An easy computation shows that if  $\phi : H \rightarrow S^1$  is a character, then  $\phi(x, y, z) = \phi(x, y)$ . Thus the one dimensional representations of a Heisenberg odometer are determined entirely by the eigenvalues of its associated  $\mathbb{Z}^2$  odometer. We describe below the eigenvalues of a  $\mathbb{Z}^2$  odometer on  $\overleftarrow{\mathbb{Z}^2/A_n\mathbb{Z}^2}$  in terms of the sequence of matrices  $A_n$  and using the results of Section 4 we identify explicitly those finite dimensional, irreducible representations of the Heisenberg group which arise in the spectral decomposition of the  $H$  odometer action on  $\overleftarrow{H/\Gamma_n}$  as a function of the triples  $(A_n, m_n, i_n)$  defining the subgroups  $\Gamma_n$ .

**5.1. Discrete spectrum of general odometer actions.** Let  $G$  be a discrete, finitely generated, and residually finite group. Fix a  $G$  odometer on  $\overleftarrow{G/\Gamma_n}$ . Let  $U : G \times L^2(\overleftarrow{G/\Gamma_n}, \mu) \rightarrow L^2(\overleftarrow{G/\Gamma_n}, \mu)$  denote the induced unitary operator of the odometer action defined by

$$(15) \quad U(g)(f(\mathbf{x})) = f(g^{-1}\mathbf{x}).$$

**Theorem 5.1.**  *$U$  admits a decomposition into finite dimensional irreducible representations of  $G$  of the form*

$$U = \bigoplus_{k=1}^{\infty} U_k$$

*with the property that there exists an increasing sequence  $k_n \rightarrow \infty$  so that for all  $n$ ,  $\bigoplus_{k=1}^{k_n} U_k$  is equivalent to the unique irreducible decomposition of the regular representation of  $G/\Gamma_n$ .*

Before proving the result, we introduce some additional notation. Let  $\Sigma$  be the Borel  $\sigma$ -algebra on  $\overleftarrow{G/\Gamma_n}$  generated by all cylinder sets  $[n; \gamma]$ ,  $n \geq 1$  and  $\gamma \in G/\Gamma_n$ ; for each  $n \geq 1$ , let  $\Sigma_n$  be the  $\sigma$ -algebra generated by the  $n^{\text{th}}$ -stage cylinder sets, and let  $\mu_n = \mu|_{\Sigma_n}$  be the induced probability measure on the cylinder sets, which is normalized counting measure. Let  $X_n = (\overleftarrow{G/\Gamma_n}, \Sigma_n, \mu_n)$ . If  $g \in \Gamma_n$ , then  $U(g)(\chi_{[n; \gamma]}) = \chi_{[n; g\gamma]} = \chi_{[n; \gamma]}$ , so the representation  $U$  induces a representation  $U^{(n)}$  of  $G/\Gamma_n$  on  $L^2(X_n)$ . By using the natural unitary isomorphism between the finite dimensional spaces  $L^2(G/\Gamma_n)$  and  $L^2(X_n)$  given by  $\chi_\gamma \mapsto \chi_{[n; \gamma]}$ , one easily checks that  $U^{(n)}$  is equivalent to the regular representation of  $G/\Gamma_n$  over  $L^2(G/\Gamma_n)$ .

This equivalence allows us to use the machinery of regular representations of finite dimensional groups to prove Theorem 5.1 constructively. In particular we will use the following two classical results which we state in our context.

**Theorem 5.2.** (Maschke's Theorem)[5] *For every  $n$ ,*

$$L^2(X_n) = \bigoplus_{1 \leq k \leq k_n} \mathfrak{F}_k^n$$

*where each  $\mathfrak{F}_k^n$  is an irreducible,  $U^{(n)}$  invariant subspace. Furthermore, suppose that  $V \subset L^2(X_n)$  is a  $U^{(n)}$  invariant subspace. Then  $L^2(X_n)$  admits a decomposition of the form*

$$V \oplus \mathfrak{F}_{k_1}^n \oplus \cdots \oplus \mathfrak{F}_{k_j}^n$$

*for some sub collection of the subspaces  $\mathfrak{F}_k^n$ .*

It is obvious that  $L^2(X_n) \subset L^2(X_{n+1})$ . Moreover, a compatibility relation exists between the  $G/\Gamma_n$ -representation,  $U^{(n)}$  and the  $G/\Gamma_{n+1}$ -representation,  $U^{(n+1)}$ .

**Lemma 5.3.** *For every  $\gamma \in G/\Gamma_{n+1}$ ,  $U^{(n+1)}(\gamma)$  restricted to  $L^2(X_n)$  coincides with  $U^{(n)}(\pi_n(\gamma))$ .*

*Proof.* It is sufficient to verify the condition for a given  $\bar{\gamma} \in G/\Gamma_n$  and the associated characteristic function  $\chi_{[n;\bar{\gamma}]}$ :

$$\begin{aligned} U^{(n)}(\pi_n(\gamma))\chi_{[n;\bar{\gamma}]} &= \chi_{[n;\pi_n(\gamma)\bar{\gamma}]} = \sum_{\hat{\gamma} \in \pi_n^{-1}(\pi_n(\gamma)\bar{\gamma})} \chi_{[n+1;\hat{\gamma}]} \\ &= \sum_{\hat{\gamma} \in \gamma\pi_n^{-1}(\bar{\gamma})} \chi_{[n+1;\hat{\gamma}]} = \sum_{\tilde{\gamma} \in \pi_n^{-1}(\bar{\gamma})} \chi_{[n+1;\gamma\tilde{\gamma}]} \\ &= U^{(n+1)}(\gamma)\chi_{[n;\bar{\gamma}]} \end{aligned}$$

Here we used the fact that  $\pi_n^{-1}(\pi_n(\gamma)\bar{\gamma}) = \gamma\pi_n^{-1}(\bar{\gamma})$  which follows from the definition of  $\pi_n$ . Indeed,

$$\begin{aligned} \hat{\gamma} \in \pi_n^{-1}(\pi_n(\gamma)\bar{\gamma}) &\Leftrightarrow \pi_n(\hat{\gamma}) = \pi_n(\gamma)\bar{\gamma} \Leftrightarrow \bar{\gamma} = \pi_n(\gamma^{-1})\pi_n(\hat{\gamma}) \\ &\Leftrightarrow \gamma^{-1}\hat{\gamma} \in \pi_n^{-1}(\bar{\gamma}) \Leftrightarrow \hat{\gamma} \in \gamma\pi_n^{-1}(\bar{\gamma}). \end{aligned}$$

□

Lemma 5.3 shows that  $L^2(X_n) \subset L^2(X_{n+1})$  is a  $U^{(n+1)}$  invariant subspace therefore by Theorem 5.2 we have a decomposition

$$L^2(X_{n+1}) = L^2(X_n) \oplus \mathfrak{F}_{k_{n+1}}^{n+1} \oplus \cdots \oplus \mathfrak{F}_{k_{n+1}}^{n+1}$$

into  $U^{(n+1)}$  (and therefore  $U$ ) invariant subspaces.

Lemma 5.3 also implies that the decomposition of  $L^2(X_n)$  into  $U^{(n)}$  invariant irreducible subspaces is a decomposition into  $U^{(n+1)}$  invariant and irreducible subspaces, therefore

$$\mathfrak{F}_1^n \oplus \cdots \oplus \mathfrak{F}_{k_n}^n \oplus \mathfrak{F}_{k_{n+1}}^{n+1} \oplus \cdots \oplus \mathfrak{F}_{k_{n+1}}^{n+1}$$

is a decomposition of  $L^2(X_{n+1})$  into  $U^{(n+1)}$  (and thus  $U$ ) invariant, irreducible, finite dimensional subspaces.

So for all  $n$ , by relabelling, we have the decomposition  $L^2(X_n) = \bigoplus_{1 \leq k \leq k_n} \mathfrak{F}_k$  such that  $U_k = U|_{\mathfrak{F}_k}$  is an irreducible, finite dimensional representation of  $\bar{G}$ . Recalling the fact that  $U^{(n)}$  is equivalent to the regular representation of  $G/\Gamma_n$ , the collection  $U_k$  for  $k = 1, \dots, k_n$  must include all representations of that finite group.

To complete the argument note that the space  $\mathfrak{V} = \bigoplus_{k=1}^{\infty} \mathfrak{F}_k$  contains all characteristic functions  $\chi_{[n;\gamma]}$ . Furthermore, the collection of characteristic functions generates an algebra of continuous functions that separates points in  $\overleftarrow{G/\Gamma_n}$ , is closed under complex conjugation and contains the constants. Therefore, by the Stone-Weierstrass Theorem  $\mathfrak{V}$  contains the continuous functions on  $\overleftarrow{G/\Gamma_n}$ , and by the density of the continuous functions in the  $L^2$  norm, must therefore be all of  $L^2(\overleftarrow{G/\Gamma_n})$ .

**Corollary 5.4.** *Every  $G$  sub-odometer has discrete spectrum.*

*Proof.* In [3] the authors prove that every sub-odometer is the factor of an odometer. The result then follows from Theorem 5.1. □

If in addition we suppose that  $G$  is a discrete countable amenable group, recent developments in the field yield the following immediate corollary.

**Corollary 5.5.** *G odometer actions have zero entropy.*

*Proof.* Suppose not. It would then follow from [9] that  $(\overleftarrow{G/\Gamma_n}, \mu, G)$  has a Bernoulli factor. By [1] this factor would have countable Lebesgue spectrum, contradicting Theorem 5.1.  $\square$

**5.2. Spectrum of  $\mathbb{Z}^2$  odometers.** Let us fix a  $\mathbb{Z}^2$  odometer on  $\overleftarrow{\mathbb{Z}^2/A_n\mathbb{Z}^2}$ . Let

$$\phi(x, y) = e^{2\pi i(\alpha x + \xi y)}$$

denote a character of  $\mathbb{Z}^2$ . By an abuse of notation we refer to the pair  $(\alpha, \xi)$ , rather than the associated character, as an eigenvalue of the action. We further only concern ourselves with  $(\alpha, \xi) \in \mathbb{T}^2$ .

An easy computation shows that  $(\alpha, \xi) \in \mathbb{T}^2$  is an eigenvalue for this odometer action if and only if for some  $n$

$$(16) \quad (\alpha, \xi) \in (A_n^T)^{-1}\mathbb{Z}^2.$$

It is therefore clear that there are  $\det(A_n)$  eigenvalues which are associated to the  $n$ th stage of the odometer. In what follows we fix a stage  $n$  and for ease of notation we suppress the subscript  $n$ .

Suppose  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Then from (16) it follows that  $(\alpha, \xi)$  is an eigenvalue if and only if it has the form

$$(17) \quad \left( \frac{(du - cv)}{\det A}, \frac{(-bu + av)}{\det A} \right)$$

for some  $(u, v) \in \mathbb{Z}^2$  where each component is considered  $\pmod{1}$ . There are, then  $\frac{\det A}{\gcd(c, d)}$  distinct possible choices for  $\alpha$  and  $\frac{\det A}{\gcd(a, b)}$  distinct possible choices for  $\xi$ .

If  $b = c = 0$  then it is clear that there will be  $ad$  pairs of the form  $(\frac{j}{a}, \frac{j'}{d})$ ,  $j \in \{0, \dots, a-1\}$ ,  $j' \in \{0, \dots, d-1\}$ . Another easy observation is that if the rows of  $A$  have relatively prime entries, i.e.  $\gcd(a, b) = \gcd(c, d) = 1$  then given  $\alpha$  there can only be a unique  $\xi$  such that  $(\alpha, \xi)$  is an eigenvalue. This follows from the fact that if  $\gcd(c, d) = 1$ , then in (17) the numerator of the first component can be made to take any value  $0, \dots, \det A - 1$ , taken  $\pmod{\det A}$ . There are then exactly  $\det A$  choices for  $\alpha$ , each with a corresponding  $\xi$ . But there are  $\det A$  pairs possible, so this exhausts all possible pairs that can occur, meaning all possible  $\xi$  have now appeared.

Notice in the above two examples, for any  $\alpha$  that appears in the spectrum, the number of  $\xi$  that can pair with it is the same for any  $\alpha$ . It will follow from our work below that this holds in general. In particular, given any matrix  $A$  we can decompose it as:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Delta \cdot \hat{A} = \begin{pmatrix} \gcd(a, b) & 0 \\ 0 & \gcd(c, d) \end{pmatrix} \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}.$$

What we will show is that the eigenvalues of the action coming from  $A$  are related to the eigenvalues coming from  $\hat{A}$  in a nice geometric fashion. Start with a pair  $(\hat{\alpha}, \hat{\xi})$  which is an eigenvalue for the  $\mathbb{Z}^2$  odometer on  $\overleftarrow{\mathbb{Z}^2/\hat{A}_n\mathbb{Z}^2}$  from stage  $n$ , but for technical reasons we ask that  $\hat{\alpha}, \hat{\xi} \in (0, 1]$ . A simple computation shows that  $(\hat{\alpha}, \hat{\xi}) \in$

$(A^T)^{-1}$  implies that for  $j \in \{0, \dots, \gcd(a, b) - 1\}$  and  $j' \in \{0, \dots, \gcd(c, d) - 1\}$

$$\begin{pmatrix} \frac{j}{\gcd(a, b)} & 0 \\ 0 & \frac{j'}{\gcd(c, d)} \end{pmatrix} (\hat{\alpha}, \hat{\xi}) \in \left( \begin{pmatrix} \frac{j}{\gcd(a, b)} & 0 \\ 0 & \frac{j'}{\gcd(c, d)} \end{pmatrix} \hat{A}^T \right)^{-1}$$

and therefore the pair  $(\frac{j}{\gcd(a, b)}\hat{\alpha}, \frac{j'}{\gcd(c, d)}\hat{\xi})$  must be an eigenvalue of the odometer on  $\overleftarrow{\mathbb{Z}^2/A_n\mathbb{Z}^2}$  from stage  $n$ . There are exactly  $\gcd(a, b) \cdot \gcd(c, d) \cdot \det \hat{A} = \det(A)$  such pairs so these are in fact all the eigenvalues corresponding to stage  $n$ .

So given any  $\mathbb{Z}^2$  odometer, there is now a clear algorithm for listing the corresponding eigenvalues corresponding to a fixed stage  $n$ . Using (17) first find  $(u, v) \in \mathbb{Z}^2$  such that

$$(18) \quad \hat{d}u - \hat{c}v = 1$$

then the following is a complete list of the eigenvalues of the  $\mathbb{Z}^2$  action from this stage:

$$(19) \quad \left\{ \left( \frac{qj}{\gcd(a, b)} \frac{\hat{d}u - \hat{c}v}{\det \hat{A}}, \frac{qj'}{\gcd(c, d)} \frac{-\hat{b}u + \hat{a}v}{\det \hat{A}} \right) : \right. \\ \left. q = 0, \dots, \det \hat{A} - 1, j = 0, \dots, \gcd(a, b) - 1, k = 0, \dots, \gcd(c, d) - 1 \right\}.$$

Note that since  $A_n\mathbb{Z}^2 < A_k\mathbb{Z}^2$  for all  $k < n$ , (19) will include all previous stage eigenvalues as well.

**5.3. Spectral analysis of Heisenberg odometers.** Indukaev [6] shows that all  $p$  dimensional, irreducible representations of the discrete Heisenberg group  $H$  have the following structure:

$$(20) \quad U_{(x, y, z)} : \epsilon_j \mapsto e^{2\pi i(y\xi + (z + jy)\eta + [\frac{x+j}{p}]\alpha)} \epsilon_{(j-x) \bmod p}$$

where  $\eta \in \mathbb{T}^1$  is an irreducible fraction of the form  $\frac{\ell}{p}$ ,  $(\xi, \alpha) \in \mathbb{T}^2$  is arbitrary and  $\epsilon_j$  denotes the projection onto the  $j$ th coordinate in a finite  $p$ -dimensional space,  $j = 0, \dots, p - 1$ .

Fix a Heisenberg odometer action on  $\overleftarrow{H/\Gamma_n}$  and let  $(A_n, m_n, i_n)$  denote the triples associated with the subgroups  $\Gamma_n$ . Here we determine the triples  $(\alpha, \xi, \eta)$  that can occur in the spectral decomposition of the odometer action as a function of  $(A_n, m_n, i_n)$ . Recall from Proposition 3.2 that it is necessary that  $m_n$  divides all the entries of  $A_n$  for all  $n$ . We consider here only the case where  $m_n$  is the greatest common divisor of the entries of  $A_n$ . An easy application of Lemma 4.3 shows that any other choice for the sequence  $m_n$  will result in a factor of this odometer, therefore there will be no new representations that can arise in their spectral decompositions.

Several easy observations follow from our proof of Theorem 5.1. First, since there is an inductive structure to the spectral decomposition of  $L^2(\overleftarrow{H/\Gamma_n}, \mu)$  it suffices to describe the choices of  $(\alpha, \xi, \eta)$  that arise in the decomposition of  $L^2(H/\Gamma_n)$  for some  $n$ . It also follows from the proof that the representations that occur must in fact come from the regular representations of the subgroups  $H/\Gamma_n$ . In particular, for  $(x, y, z) \in \Gamma_n$ , (20) must reduce to the identity operator, meaning that

$$(21) \quad y\xi + (z + jy)\frac{\ell}{p} + \left[ \frac{x + j}{p} \right] \alpha \in \mathbb{Z}.$$

If (21) holds, then the general theory of regular representations of finite groups implies that the triple  $(\alpha, \xi, \frac{\ell}{p})$  must occur in the spectral decomposition of the odometer.

Since  $\Gamma_n$  contains elements of the form  $(0, 0, z)$  we see that (21) reduces to requiring that  $\frac{z\ell}{p} \in \mathbb{Z}$  for any  $z \in m_n\mathbb{Z}$ , which implies that  $p$  must divide  $m_n$ .

If  $p = 1$ , then in (21) we have  $j = 0, \ell = 1$  and since  $z \in \mathbb{Z}$  the one dimensional representations that occur have the form  $e^{2\pi i(y\xi + x\alpha)}$ . Thus, as we noted before, we see that the one dimensional representations that occur in the decomposition are precisely those that come from pairs  $(\alpha, \xi)$  that are eigenvalues of the associated  $\mathbb{Z}^2$  odometer.

We now fix a stage  $n$  and some  $p$  which divides  $m_n$ , an  $\ell \in 1, \dots, p-1$  such that  $\gcd(\ell, p) = 1$ . For ease of notation in what follows we suppress the subscript  $n$  where there is no danger of ambiguity.

Notice that  $[\frac{x+j}{p}] = \frac{x}{p}$  and  $\frac{jy}{p} \in \mathbb{Z}$ , so for  $(x, y, z) \in \Gamma_n$  (21) reduces to requiring

$$y\xi + \frac{z\ell}{p} + \frac{x}{p}\alpha = y\xi + \frac{(sm + i(x, y))\ell}{p} + \frac{x}{p}\alpha \in \mathbb{Z}$$

for all  $s \in \mathbb{Z}$ . Again using the fact that  $p$  is a divisor of  $m$  we are left with

$$y\xi + \frac{i(x, y)\ell}{p} + \frac{x}{p}\alpha \in \mathbb{Z}.$$

In the case of a flat odometer, without loss of generality we can assume  $i_n(x, y) = 0$  for all  $n$ , and we can conclude that the triples that occur are exactly those  $(\alpha, \xi, \frac{\ell}{p})$  where  $(\frac{\alpha}{p}, \xi)$  is an eigenvalue of the associated  $\mathbb{Z}^2$  odometer. Otherwise the values of  $\xi$  and  $\alpha$  depend on the structure of  $i$ . Below we compute some examples to show the effect of  $i$  on the triples that can appear.

Consider the sequence of matrices  $A_n = \begin{pmatrix} k_n & 0 \\ 0 & k_n \end{pmatrix}$  and  $m_n = k_n$  where the sequence  $k_n$  as defined in Example 4.8. For any choice of the sequence  $i_n$  for which the triple  $(A_n, m_n, i_n)$  defines a Heisenberg odometer, the associated  $\mathbb{Z}^2$  odometer has eigenvalues of the form  $(\frac{j}{k_n}, \frac{j'}{k_n})$  for  $j, j' \in \{0, \dots, k_n - 1\}$ . Fix a stage  $n$ , assume  $p$  divides  $k_n$  and that  $\ell$  is relatively prime to  $p$ . We consider three different choices for the sequence  $i_n$ .

- (a) Let  $i_n = 0$ . The  $p$ -dimensional representations that occur will be those corresponding to the triples:

$$\left( \frac{pj}{k_n}, \frac{j'}{k_n}, \frac{\ell}{p} \right).$$

- (b) Let  $i_n = \frac{x}{k_n}$ . This gives rise to the odometer constructed in Example 4.8 where we showed that it is not flat so we expect a different set of representations to occur. Indeed, the triples  $(\alpha, \xi, \frac{\ell}{p})$  that can occur now have to satisfy:

$$\begin{aligned} y\xi + \frac{i(x, y)\ell}{p} + \frac{x}{p}\alpha &= y\xi + \frac{x\ell}{pk_n} + \frac{x}{p}\alpha = y\xi + x\left(\frac{\ell}{pk_n} + \frac{\alpha}{p}\right) \\ &= y\xi + x\left(\frac{\ell + k_n\alpha}{pk_n}\right) \in \mathbb{Z}. \end{aligned}$$

Therefore the representations that occur will be those corresponding to the triples

$$\left(\frac{jp - \ell}{k_n}, \frac{j'}{k_n}, \frac{\ell}{p}\right).$$

- (c) Let  $i_n = \frac{y}{k_n}$ . An argument parallel to that given in Example 4.8 shows that this choice of  $i_n$  gives rise to an odometer action which is not flat. The triples that occur must satisfy

$$y\xi + \frac{y\ell}{k_np} + \frac{x}{p}\alpha = y\left(\xi + \frac{\ell}{k_np}\right) + x\frac{\alpha}{p} \in \mathbb{Z}$$

which gives us triples of the form

$$\left(\frac{pj}{k_n}, \frac{j'p - \ell}{k_np}, \frac{\ell}{p}\right).$$

Thus, this last example is not flat and also not conjugate to the odometer from Example 4.8.

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